Recall the Solovay-Kitaev algorithm:

function Solovay-Kitaev(Gate U, depth n)
if $(n == 0)$
 Return Basic Approximation to U
else
→ Set U_{n-1} = Solovay-Kitaev$(U, n-1)$
→ Set V, W = GC-Decompose$(UU_{n-1}^{\dagger})$
→ Set V_{n-1} = Solovay-Kitaev$(V, n-1)$
→ Set W_{n-1} = Solovay-Kitaev$(W, n-1)$
→ Return $U_n = V_{n-1} W_{n-1} V_{n-1}^{\dagger} W_{n-1}^{\dagger} U_{n-1}$;

Analysis of runtime:

$l_n \equiv$ length of instruction sequence returned by SK-algorithm
$t_n \equiv$ corresponding runtime

From the algorithm we read off:

$$\varepsilon_n = c_{appr}\, \varepsilon_{n-1}^{3/2} \quad \text{(to be proven)}$$
$$l_n = 5 l_{n-1} \quad \text{(last line)}$$
$$t_n \leq \underbrace{3 t_{n-1}}_{\text{3 recursive calls}} + \underbrace{\text{const}}_{\text{finding group commutator}}$$

We compute

(1) $$\varepsilon_n = c\,\varepsilon_{n-1}^{3/2} = \frac{1}{c^2}\left(c^2 \varepsilon_{n-1}\right)^{3/2} = \frac{1}{c^2}\left(c^2\varepsilon_0\right)^{\left(\frac{3}{2}\right)^n}$$

(2) $$l_n = O(5^n)$$

(3) $$t_n = O(3^n)$$

Inverting (1) gives n in terms of ε:

$$\log\left(\varepsilon_n c^2\right) = \log\left(c^2\varepsilon_0\right)\left(\frac{3}{2}\right)^n$$

$$\Leftrightarrow \quad \frac{\log\left(\varepsilon_n c^2\right)}{\log\left(c^2\varepsilon_0\right)} = e^{n\log\frac{3}{2}}$$

$$\Leftrightarrow \quad \log\left[\frac{\log\left(\varepsilon_n c^2\right)}{\log\left(c^2\varepsilon_0\right)}\right] = n\log\frac{3}{2}$$

$$\Leftrightarrow \quad n = \left\lceil \frac{\log\left[\frac{\log\left(\varepsilon c^2\right)}{\log\left(\varepsilon_0 c^2\right)}\right]}{\log\frac{3}{2}} \right\rceil$$

Inserting back into (2) and (3) gives:

$$l_\varepsilon = \exp\left(\log 5 \, \frac{\log\left(\frac{\log\left(c^2\varepsilon\right)}{\log\left(\varepsilon_0 c^2\right)}\right)}{\log 3/2}\right)$$

$$= \left(\frac{\log(c^2 \varepsilon)}{\log(c^2 \varepsilon_0)} \right)^{\log 5 / \log 3/2}$$

$$= O\left(\log\left(\frac{1}{\varepsilon}\right)^{\log 5 / \log\left(\frac{3}{2}\right)} \right) = O\left(\log\left(\frac{1}{\varepsilon}\right)^{3.97}\right)$$

$$t_\varepsilon = O\left(\log\left(\frac{1}{\varepsilon}\right)^{\log 3 / \log\left(\frac{3}{2}\right)} \right) = O\left(\log\left(\frac{1}{\varepsilon}\right)^{2.71}\right)$$

$\longrightarrow$ runtime and sequence length scale polylogarithmically with accuracy

Proof of details:

Step 1) : Balanced commutators in $SU(2)$

Suppose $U \in SU(2)$ satisfies $d(I, U) < \varepsilon$

goal: find V and W with $VWV^\dagger W^\dagger = U$ and $d(I, V), d(I, W) < c_{gc} \sqrt{\varepsilon}$

To find V and W, we examine a special case:

$$V \equiv e^{-i\phi X/2} = \begin{pmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}$$ rotation by angle ϕ about $\hat{x}$-axis

$$W \equiv e^{-i\phi Y/2} = \begin{pmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}$$ rotation by angle ϕ about $\hat{y}$-axis

Then the resulting group commutator is a rotation about some axis $\hat{n}$, by angle θ:

$$\sin\left(\theta/2\right) = 2\sin^2(\phi/2)\sqrt{1-\sin^4(\phi/2)} \qquad (*)$$

(exercise)

Now suppose that U is a rotation by arbitrary angle Θ about arbitrary axis $\hat{p}$

solve ϕ in terms of Θ using $(*)$

$\rightarrow U = S(VWV^\dagger W^\dagger)S$

where conjugation by S rotates $\hat{n}$-axis onto $\hat{p}$-axis

$\rightarrow U = \tilde{V}\tilde{W}\tilde{V}^\dagger\tilde{W}^\dagger$

where $\tilde{V} \equiv SVS^\dagger$, $\tilde{W} \equiv SWS^\dagger$

For a unitary rotation T by an angle τ, we have $d(I,T) = 2\sin(\tau/4) = \frac{\tau}{2} + O(\tau^3)$

For U close to the identity, we get from $(*)$:

$$d(I,U) \approx 2d(I,V)^2 = 2d(I,W)^2$$

$\rightarrow U = VWV^\dagger W^\dagger$, $d(I,V) = d(I,W) \approx \sqrt{\frac{d(I,U)}{2}} < \sqrt{\frac{\varepsilon}{2}}$

$\rightarrow c_{gc} \approx \frac{1}{\sqrt{2}}$

Step 2): approximating a commutator

<u>Lemma 1:</u>

Suppose $V, W, \tilde{V}$, and $\tilde{W}$ are unitaries such that $d(V,\tilde{V})$, $d(W,\tilde{W}) < \Delta$, and also $d(I,V)$, $d(I,W) < \delta$. Then:

$$d(VWV^\dagger W^\dagger, \tilde{V}\tilde{W}\tilde{V}^\dagger\tilde{W}^\dagger) < 8\Delta\delta + 4\Delta\delta^2 + 8\Delta^2 + 4\Delta^3 + \Delta^4$$

replacing Δ by ε_{n-1}, and δ by $c_{gc}\sqrt{\varepsilon_{n-1}}$ gives:

$$d(VWV^\dagger W^\dagger, \tilde{V}\tilde{W}\tilde{V}^\dagger\tilde{W}^\dagger) < c_{appr.}\,\varepsilon_{n-1}^{3/2}$$

where $c_{appr.} \approx 8\,c_{gc}$.

<u>Proof of Lemma 1:</u>

We begin by writing

$$\tilde{V} = V + \Delta_V, \quad \tilde{W} = W + \Delta_W$$

which gives

$$\tilde{V}\tilde{W}\tilde{V}^\dagger\tilde{W}^\dagger = VWV^\dagger W^\dagger + \Delta_V W V^\dagger W^\dagger + V\Delta_W V^\dagger W^\dagger + VW\Delta_V^\dagger W^\dagger + VWV^\dagger\Delta_W^\dagger + O(\Delta^2) + O(\Delta^3) + O(\Delta^4)$$

$$\Rightarrow d(VWV^\dagger W^\dagger, \tilde{V}\tilde{W}\tilde{V}^\dagger\tilde{W}^\dagger)$$

$$< \| \Delta_V W V^\dagger W + V\Delta_W V^\dagger W^\dagger + VW\Delta_V^\dagger W^\dagger + VWV^\dagger\Delta_W^\dagger \|$$

$$+ \underbrace{6}_{= \binom{4}{2}} \Delta^2 + \underbrace{4}_{= \binom{4}{3}} \Delta^3 + \Delta^4 \qquad (**)$$

Expanding $W = I + \delta_W$ gives

$$\Delta_V W V^\dagger W^\dagger + V W \Delta_V^\dagger W^\dagger = \Delta_V V^\dagger + V \Delta_V^\dagger + O(\Delta\delta) + O(\Delta\delta^2)$$

$$\Rightarrow \| \Delta_V W V^\dagger W^\dagger + V W \Delta_V^\dagger W^\dagger \| < \| \Delta_V V^\dagger + V \Delta_V^\dagger \| + 4\Delta\delta + 2\Delta\delta^2$$

Moreover, unitarity of V and $V + \Delta_V$ gives

$$\Delta_V V^\dagger + V \Delta_V^\dagger = - \Delta_V \Delta_V^\dagger$$

$$I = (V + \Delta_V)^\dagger (V + \Delta_V)$$

$$= (V^\dagger + \Delta_V^\dagger)(V + \Delta_V) = I + V^\dagger \Delta_V + \Delta_V^\dagger V + \Delta_V^\dagger \Delta_V$$

$$\Leftrightarrow \Delta_V V^\dagger + V \Delta_V^\dagger = - \Delta_V \Delta_V^\dagger$$

$$\Rightarrow \| \Delta_V W V^\dagger W^\dagger + V W \Delta_V^\dagger W^\dagger \| < \Delta^2 + 4\Delta\delta + 2\Delta\delta^2$$

Similarly

$$\| V \Delta_W V^\dagger W^\dagger + V W V^\dagger \Delta_W^\dagger \| < \Delta^2 + 4\Delta\delta + 2\Delta\delta^2$$

Combining with $(**)$ and using the triangle identity gives the result

$\square$

Next, we want to show that $\{H, T\}$ is an instruction set for $SU(2)$

To this end define for $\hat{n} = (n_x, n_y, n_z)$ and $\theta \in \mathbb{R}$:

$$R_{\hat{n}}(\theta) \equiv \exp(-i\theta\, \hat{n}\cdot\vec{\sigma}/2)$$

$$= \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right)(n_x X + n_y Y + n_z Z)$$

$\rightarrow$ is rotation by angle θ about $\hat{n}$-axis of Bloch-sphere

<u>Proof:</u>

Consider now the gates T and HTH

$\rightarrow$ T is (up to phase) rotation by $\frac{\pi}{4}$ about $\hat{z}$-axis

HTH is rotation by $\frac{\pi}{4}$ about $\hat{x}$-axis

(exercise)

$$\rightarrow \exp\left(-i\frac{\pi}{8}Z\right)\exp\left(-i\frac{\pi}{8}X\right)$$

$$= \left[\cos\frac{\pi}{8} I - i \sin\frac{\pi}{8} Z\right]\left[\cos\frac{\pi}{8} I - i \sin\frac{\pi}{8} X\right]$$

$$= \cos^2\frac{\pi}{8} I - i\left[\cos\frac{\pi}{8}(X + Z) + \sin\frac{\pi}{8} Y\right]\sin\frac{\pi}{8}$$

$\rightarrow$ rotation about axis $\vec{n} = \left(\cos\frac{\pi}{8}, \sin\frac{\pi}{8}, \cos\frac{\pi}{8}\right)$

by angle $\cos\left(\frac{\theta}{2}\right) \equiv \cos^2\frac{\pi}{8}$

Can show θ is irrational multiple of 2π

Define $\theta_k = (k\theta) \bmod 2\pi \in [0, 2\pi)$

For $\delta > 0$ set $N = \lceil \frac{2\pi}{\delta} \rceil \in \mathbb{N}$

Then $\exists\ j \neq k \in \{1, \dots, N\} : |\theta_j - \theta_k| \leq \frac{2\pi}{N} < \delta$

$\Rightarrow |\theta_{k-j}| < \delta$

$\rightarrow$ sequence $\theta_{\ell(k-j)}$ fills up interval $[0, 2\pi)$ as ℓ is varied

It follows that for any $\varepsilon > 0$ and any $\alpha \in [0, 2\pi)$ $\exists\ n \in \mathbb{N}$ such that

$$d(R_{\hat{n}}(\alpha), R_{\hat{n}}(\theta)^n) < \frac{\varepsilon}{3}$$

(follows from $d(R_{\hat{n}}(\alpha), R_{\hat{n}}(\alpha+\beta)) = |1 - e^{i\beta/2}|$)

Moreover, we have

$$H R_{\hat{n}}(\alpha) H = R_{\hat{m}}(\alpha) \quad , \quad \hat{m} = (\cos\tfrac{\pi}{8}, -\sin\tfrac{\pi}{8}, \cos\tfrac{\pi}{8})$$

$$\rightarrow \quad d(R_{\hat{m}}(\alpha), R_{\hat{m}}(\theta)^n) < \frac{\varepsilon}{3}$$

Since any unitary U can be represented as

$$U = R_{\hat{n}}(\beta) R_{\hat{m}}(\gamma) R_{\hat{n}}(\delta) \quad \text{(exercise)}$$

we get $d(U, R_{\hat{n}}(\theta)^{n_1} H R_{\hat{n}}(\theta)^{n_2} H R_{\hat{n}}(\theta)^{n_3}) < \varepsilon$ $\square$